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OSCILLATIONS OF MECHANICAL SYSTEMS THAT DO NOT BECOME LINEAR WHEN THE PARAMETER VANISHES

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V. T. GRUMONDZ
(Moscow)

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The results of investigations in [1] are extended to multidimensional systems that become nonlinear at $\mu = 0$. Two-dimensional mechanical systems were investigated in [2, 3]. The characteristic equations of systems considered here contain in the critical system either a pair of pure imaginary roots or two zero roots with one or two groups of solutions and n roots with negative real parts in the adjoint system. It is shown that the investigation of such systems necessitates the imposition on the system of some constraints that supplement those specified in [1]. The auxilliary function $u_k^{(1)}(\theta)$ used in the determination of Liapunov's function is derived by a different method than in [1 — 3]. In two of the three investigated cases the problem is reduced to the determination of roots of some integral real irrational function. An example is presented.

The question of existence and stability within region N of steady oscillations of multidimensional quasi-linear mechanical systems whose equations of motion written in the standard Cauchy form contain in their right-hand sides series in powers of some positive parameter μ. For $\mu = 0$ the system becomes linear and its characteristic equation has a pair of pure imaginary roots and n roots with negative real parts. An algorithm for the determination of region N is also indicated.

1. Let us consider a mechanical system whose differential equations in the standard Cauchy form are

$$x^\cdot = X_0(x, y) + \sum_{l=1}^{\infty} \mu^l X_l(x, y, x_1, \ldots, x_n) \tag{1.1}$$

$$y^\cdot = Y_0(x, y) + \sum_{l=1}^{\infty} \mu^l Y_l(x, y, x_1, \ldots, x_n)$$

$$x_j^\cdot = \sum_{k=1}^{n} p_{jk} x_k + \sum_{l=1}^{\infty} \mu^l X_{jl}(x, y, x_1, \ldots, x_n), \quad j = 1, 2, \ldots, n$$

whose characteristic system converts for $\mu = 0$ into the following nonlinear system:

$$x^\cdot = X_0(x, y),\ y^\cdot = Y_0(x, y) \tag{1.2}$$

We assume that all of the 1° —4° stipulations in [1] are satisfied by (1.1), i.e. the right-hand sides of (1.1) are absolutely convergent series in the investigated region U of variation of variables x, y, x_j and parameter μ, X_l, Y_l, X_{jl} are sums of forms relative to variables $x, y, x_1, \ldots, x_n$ of any finite order $\nu_{xl}, \nu_{yl}, \nu_{jl}$ with constant coefficients and lower powers of form $\eta_{xl}, \eta_{yl}, \eta_{jl}$ are higher than unity, the roots of polynomial $D(\varkappa) = |p_{ij} - \delta_{ij}\varkappa|$ are different and have negative real parts, and the right-hand sides of the critical system vanish when $x = y = 0$. Note that when the last condition is not satisfied by (1.1), then unlike in [1], it can be reduced to the required form only if $h_1\varkappa_1 + \ldots + h_n\varkappa_n \neq 0$ for any nonnegative integers h_k such that $\Sigma h_k > 0$ [4]. We also assume that the following conditions are satisfied by (1.1).

1°. Functions X_0 and Y_0 are of the form

$$X_0(x, y) = \sum_{\alpha+\beta=m_0} a_{\alpha\beta}x^\alpha y^\beta, \quad Y_0(x, y) = \sum_{\alpha+\beta=m_0} b_{\alpha\beta}x^\alpha y^\beta, \quad m_0 > 0 \tag{1.3}$$

where $a_{\alpha\beta}$, and $b_{\alpha\beta}$ are constant coefficients.

2°. Functions $F(x, y) = xY_0 - yX_0$ and $R(x, y) = xX_0 + yY_0$ are such that $F(x, y)$ is of constant sign and $\varphi(2\pi) = 0$, where

$$\varphi(\theta) = \int_0^\theta \frac{R(\cos\psi, \sin\psi)}{F(\cos\psi, \sin\psi)}\, d\psi$$

3°. $m_1 \geqslant m_0$, where $m_1 = \min\{m_{x_1}, m_{y_1}\}$, m_{xl}, m_{yl} are the lowest powers of forms that in the critical system are free of $x_1, \ldots, x_n$.

We transform (1.1) using the same method as in [1] to the canonical form, carry out the substitutions $x = \bar{r}\cos\theta$ and $y = \bar{r}\sin\theta$, and instead of (1.1) obtain the following system:

$$\bar{r}^\cdot = \bar{R}_0(\bar{r}, \theta) + \sum_{l=1}^{\infty} \mu^l \bar{R}_l(\bar{r}, \theta, z_1, \ldots, z_n) \tag{1.4}$$

$$\theta^\cdot = \bar{F}_0(\bar{r}, \theta) + \sum_{l=1}^{\infty} \mu^l \bar{F}_l(\bar{r}, \theta, z_1, \ldots, z_n)$$

$$z_k^\cdot = g_k z_k - h_k z_{T+k} + \sum_{l=1}^{\infty} \mu^l \bar{Z}_{kl}(\bar{r}, \theta, z_1, \ldots, z_n)$$

$$\dot{z}_{k+T} = h_k z_k + g_k z_{T+k} + \sum_{l=1}^{\infty} \mu^l \bar{Z}_{k+T,\,l}(\bar{r}, \theta, z_1, \ldots, z_n)$$

$$z_s^\cdot = d_s z_s + \sum_{l=1}^{\infty} \mu^l \bar{Z}_{sl}(\bar{r}, \theta, z_1, \ldots, z_n), \quad k = 1, 2, \ldots, T$$

$$s = 2T + 1, \ldots, n$$

$$\bar{R}_0(\bar{r}, \theta) = X_0(\bar{r}\cos\theta, \bar{r}\sin\theta)\cos\theta + Y_0(\bar{r}\cos\theta, \bar{r}\sin\theta)\sin\theta$$

$$\bar{F}_0(\bar{r}, \theta) = \frac{1}{r}[Y_0(\bar{r}\cos\theta, \bar{r}\sin\theta)\cos\theta - X_0(\bar{r}\cos\theta, \bar{r}\sin\theta)\sin\theta]$$

When the coefficients of transformation to the canonical form [1] are known, functions $\bar{R}_l$, $\bar{F}_l$, $\bar{Z}_{\alpha l}$ can be readily determined.

To solve the problem of existence and stability relative to the region of oscillations in system (1.4) that are stable in the sense given in [2] we shall try to use the results of investigations [1]. For this it is, first of all, necessary to transform (1.4) so as to eliminate the nonlinear functions $\bar{R}_0$ and $\bar{F}_0$ in the right-hand sides of the critical system. We proceed as follows. Noting that system (1.2) in terms of variables $\bar{r}$, θ with conditions 1° and 2° satisfied has the periodic solution $\bar{r} = \bar{r}_0 e^{\varphi(\theta)}$, where $\bar{r}_0$ is an arbitrary constant, we carry out in (1.4) the substitution

$$\bar{r} = re^{\varphi(\theta)} \tag{1.5}$$

where r is the new variable. The critical system now assumes the form

$$r^{\cdot} = \sum_{l=1}^{\infty} \mu^l R_l(r, \theta, z_1, \ldots, z_n), \quad \theta^{\cdot} = F_0 + \sum_{l=1}^{\infty} \mu^l F_l(r, \theta, z_1, \ldots, z_n) \tag{1.6}$$

$$R_l = e^{-\varphi(\theta)}\bar{R}_l - rf(\theta)\bar{F}_l, \quad f(\theta) = \frac{R(\cos\theta, \sin\theta)}{F(\cos\theta, \sin\theta)}$$

$$F_0(r, \theta) = \frac{1}{r^2} e^{-2\varphi(\theta)} F(re^{\varphi(\theta)}\cos\theta, re^{\varphi(\theta)}\sin\theta)$$

$$F_l(r, \theta, z_1, \ldots, z_n) = \bar{F}_l(re^{\varphi(\theta)}, \theta, z_1, \ldots, z_n)$$

The form of the adjoint system remains unchanged. For μ^l functions $Z_{\alpha l}(r, \theta, z_1, \ldots, z_n)$ are obtained from $\bar{Z}_{\alpha l}(\bar{r}, \theta, z_1, \ldots, z_n)$ as the result of substitution (1.5). Functions R_l, F_l, $Z_{\alpha l}$ are the sums of forms relative to $r, z_1, \ldots, z_n$ with coefficients that are 2π-periodic functions of θ, hence their structure is the same as that of the corresponding functions in [1]. Functions (1.6) are exactly the same as functions R_l and F_l in [1], if we set $f(\theta) \equiv 0$. Function $F_0(r, \theta) = r^{m_0-1}F_0^{(m_0-1)}(\theta)$ is of fixed sign according to assumption 2°, hence $F_0^{(m_0-1)} \neq 0$ for all $\theta \in [0, 2\pi)$, $r > 0$.

We introduce the new variable ρ by formula

$$r = \rho + \mu\sum_{q=1}^{M_1} \rho^q u_1^{(q)}(\theta), \quad M_1 = \max\{M_{x1}, M_{y1}\} \tag{1.7}$$

where M_{x1} and M_{y1} are the highest orders of forms appearing in X_1 and Y_1 and independent of $z_1, \ldots, z_n$, $u_1^{(q)}(\theta)$ are some, so far unknown, 2π-periodic functions, and μ is assumed so small throughout the investigation region U that function ρ which is the solution of (1.7) is positive definite for all $r > 0$ and any θ. It is further assumed that under the above conditions $r > 0$ and $H = \partial r(\rho, \theta)/\partial\rho > 0$. We select functions $u_1^{(q)}$, $u_1^{(q)}(0) = 0$ so as to have them 2π-periodic and, as the result of transformation (1.7), that the expression at μ in the right-hand side of the first equation of the critical system is in the form of sum of the polynomial

$$L_1(\rho) = \sum_{q=m_1}^{M_1} \rho^q g_1^{(q)}$$

with constant coefficient $g_1^{(q)}$ and of some function of the form

$$\mu R_{11} = \mu\sum_{k=0}^{N_1-1} \rho^k R_1^{(2,k)}(\theta, z_1, \ldots, z_n), \quad N_1 = \max\{\nu_{x1}, \nu_{y1}\}$$

Functions $u_1^{(q)}(\theta)$ are uniquely determined by this condition, since stipulations 1° and 2° are satisfied. Equations and their determinants yield, unlike in [1], the following solution for $u_1^{(q)}(\theta)$ and for coefficients of the polynomial $L_1(\rho)$:

$$u_1^{(1)} = u_1^{(2)} = \ldots = u_1^{(m_1-m_0)} = u_1^{(M_1-m_0+2)} = u_1^{(M_1-m_0+3)} = \ldots = u_1^{(M_1)} \equiv 0$$

$$u_1^{(s-m_0+1)} = \int_0^\theta \frac{R_1^{(1,s)}(\theta) - g_1^{(s)}}{F_0^{(m_0-1)}(\theta)} d\theta$$

$$g_1^{(m_0)} = g_1^{(m_0+1)} = \ldots = g_1^{(m_1-1)} = g_1^{(M_1+1)} = g_1^{(M_1+2)} = \ldots$$
$$g_1^{(M_1+m_0-1)} = 0$$

$$g_1^{(s)} = \left[\int_0^{2\pi} \frac{d\theta}{F_0^{(m_0-1)}(\theta)}\right]^{-1} \int_0^{2\pi} \frac{R_1^{(1,s)}(\theta)}{F_0^{(m_0-1)}(\theta)} d\theta \quad (s = m_1, \ldots, M_1)$$

where $R_1^{(1,s)}(\theta)$ are the coefficients at ρ^s of function R_1 in (1.6).

Having determined functions $u_1^{(q)}$ and constants $g_1^{(q)}$, we obtain the system of equations which consists of the critical system of the form

$$\rho^{\cdot} H = \mu L_1(\rho) + \mu R_{11}(\rho, \theta, z_1, \ldots, z_n) + \mu^2 P_2(\rho, \theta, z_1, \ldots, z_n, \mu) \tag{1.8}$$
$$\theta^{\cdot} = F_0(\rho, \theta) + \sum_{l=1}^{\infty} \mu^l F_l(\rho, \theta, z_1, \ldots, z_n)$$

and of the adjoint system derived from (1.4) using (1.5) and (1.7). Thus the investigated here system (1.1) is reduced to the form obtained in [1], whose only difference from the latter is in the form of periodic coefficients of forms whose derivation is based on transformation (1.5). Further analysis is carried out exactly as in [1]. Theorems 1 — 3 [1] on the existence of steady oscillations and their stability within region $N \subset U$, the estimate of μ, and of parameters in region N remain valid, although the estimates themselves undergo changes related to the use of transformation (1.5).

2. Let the motion of the mechanical system be defined by the system of differential equations that consists of the critical system

$$x^{\cdot} = -\lambda y + X_0 + \sum_{l=1}^{\infty} \mu^l X_l, \quad y^{\cdot} = \lambda x + Y_0 + \sum_{l=1}^{\infty} \mu^l Y_l \tag{2.1}$$

which for $\mu = 0$ becomes nonlinear of the form

$$x^{\cdot} = -\lambda y + X_0, \; y^{\cdot} = \lambda x + Y_0 \tag{2.2}$$

which has in the characteristic equation a pair of pure imaginary roots, and of the adjoint system consisting of the same n equations as (1.1). The form of functions $X_0, Y_0, X_l, Y_l, X_{sl}$ $(l = 1, 2, \ldots; s = 1, 2, \ldots, n)$ is the same as in (1.1). The investigated system will be analyzed on the same assumption as made at the beginning of Sect. 1 with respect to (1. 1), with the exception of assumptions 1° —3°. Note that, as in [1], it is always possible to reduce (2. 1) to the form in which its right-hand sides vanish when $x = y = 0$ and the stipulated conditions are satisfied [4].

We make the following additional assumptions.

1°. Functions X_0 and Y_0 are of the form (1. 3) where $m_0 \geqslant 2$, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are constant coefficients, and function $Q = \lambda\ (x^2 + y^2) + F\ (x, y)$ is positive definite throughout the investigation region U.

2°. There exists for system (2. 2) a holomorphic integral $I\ (x, y) = c^2$ of constant sign of the form $I = x^2 + y^2 + \Phi\ (x, y)$, where Φ is a function of order not lower than the third with respect to x and y, and while I retain the constant sign throughout the region of motion investigation.

It is shown in [4] that the condition of existence of such integral is the necessary and sufficient condition for (2. 2) to have periodic solutions of the form $x = x^{(1)}c + x^{(2)}c^2 + \ldots$, $y = y^{(1)}c + y^{(2)}c^2 + \ldots$, where c is an arbitrary constant and $x^{(k)}$ and $y^{(k)}$ are periodic functions. On these assumptions the equation $r^2 + \Phi$ $(r \cos\theta, r \sin\theta) = c^2$ determines according to [4] two solutions for r in the form of convergent series $r = \pm c + v_2 \cdot\ (\theta)\ c^2 \pm v_3\ (\theta)\ c^3 + \ldots$, where $v_k\ (\theta)$ is a periodic function of θ. Since the substitution of $\pi + \theta$ for θ transforms one of these solutions into the second, we take in this expression all terms with the plus sign.

As in Sect. 1, we pass to canonical variables and, then, to polar coordinates using formulas $x = \bar{r} \cos\theta$ and $y = \bar{r} \sin\theta$, and substitute r for $\bar{r}$ using formula

$$r^2 = \bar{r}^2 + \Phi\ (\bar{r} \cos\theta, \bar{r} \sin\theta) \qquad (2.3)$$

We assume that the series $\bar{r} = r + v_2\ (\theta)\ r^2 + v_3\ (\theta)\ r^3 + \ldots$ which represents the solution of (2. 3) is absolutely convergent throughout the region $U \supset N$ of solution analysis. As the result, the critical system assumes the form

$$r^\bullet = \sum_{l=1}^{\infty} \mu^l R_l (r, \theta, z_1, \ldots, z_n), \quad \theta^\bullet = \lambda + Q_0 (r, \theta) + \sum_{l=1}^{\infty} \mu^l F_l (r, \theta, z_1, \ldots, z_n) \qquad (2.4)$$

while the adjoint system, with allowance for substitution (2. 3), is of the same form as the above. The lowest with respect to r terms at each μ^s are exactly the same as the lowest terms with respect to r at μ^s of the system in Sect. 1.

We introduce the new variable ρ by formula

$$r = \rho + \mu \sum_{l=1}^{\infty} \rho^l u_1^{(l)} (\theta) \qquad (2.5)$$

where, as in Sect. 1, we assume that $u_1^{(l)}\ (\theta)$ are 2π- periodic functions and that μ is fairly small so that $r > 0$ and $H = \partial r / \partial\rho > 0$. We further assume that for all $\rho \in U$ where $U \supset N$ is the region of investigation, the series (2. 5) are absolutely convergent for each $\theta \in [0, 2\pi)$. The necessary and sufficient conditions that must be satisfied by constants $g_1^{(l)}$ if function $u_1^{(l)}$ is to be 2π- periodic, is in this case somewhat different from conditions in [1, 2] and in Sect. 1 above. Using these conditions for determining $g_1^{(l)}$ and $u_1^{(l)}$ and stipulating in addition $u_1^{(l)}(0) = 0$, we obtain:

in the case of $m_1 < m_0$

$$u_1^{(m)}(\theta) \equiv 0, \ g_1^{(m)} = 0 \ (m = 1, 2, \ldots, m_1 - 1) \tag{2.6}$$

$$u_1^{(k)}(\theta) = \frac{1}{\lambda}\int_0^\theta [R_1^{(1, k)}(\theta) - g_1^{(k)}]\, d\theta,$$

$$g_1^{(k)} = \frac{1}{2\pi}\int_0^{2\pi} R_1^{(1, k)}(\theta)\, d\theta \quad (k = m_1, \ldots, m_0 - 1)$$

$$u_1^{(i)}(\theta) = \frac{1}{\lambda}\int_0^\theta [T^{(i)}(\theta) - g_1^{(i)}]\, d\theta, \quad g_1^{(i)} = \frac{1}{2\pi}\int_0^{2\pi} T^{(i)}(\theta)\, d\theta$$

$$\left(T^{(i)}(\theta) = R_1^{(1, i)}(\theta) - \sum_{s=1}^{i-m_0+1} Q_0^{(i-s)}(\theta)\frac{du_1^{(s)}}{d\theta}, \quad i = m_0, m_0 + 1, \ldots\right)$$

where $Q_0^{(\alpha)}$ are periodic coefficients at ρ^α in function

$$Q_0\left(\rho + \sum_{l=1}^{\infty} \rho^l v_l(\theta), \theta\right)$$

in the case of $m_1 = m_0$ the same solutions, except for the equalities in the second line of (2.6);

in the case of $m_1 > m_0$ we have $u_1^{(m)} \equiv 0$, $g_1^{(m)} = 0$, and $m = 1, \ldots, m_1 - 1$, while the expressions for $u_1^{(m_1)}$, $g_1^{(m_1)}$, $u_1^{(m_1+1)}$, $g_1^{(m_1+1)}$, ... are determined by the equalities in the third line of (2.6) for $i = m_1, m_1 + 1, \ldots$.

The first equation of the transformed system is formally of the same form as the first of Eqs. (1.8), except that instead of polynomial $L_1(\rho)$ it contains the series

$$L_1^\infty(\rho) = \sum_{q=m_1}^{\infty} \rho^q g_1^{(q)}$$

Further analysis is similar to that set forth in [1], but on the assumption of absolute convergence of respective series throughout the investigation region $U \supset N$. Theorems 1 — 3 remain valid, except that in their formulation it is necessary to consider real positive zeros of odd multiplicity of the real irrational entire function $L_1^\infty(\rho)$ [5] instead of the positive roots of odd multiplicity of polynomial $L_1(\rho)$.

If the derivation of positive roots of function $L_1^\infty(\rho)$ (or of polynomial $L_1(\rho)$) proves to be impossible, the schemes considered here and in [1] admit the use of the upper and lower bounds of the disposition of every such root, which requires minimal accuracy of their determination. Thus, if

$$\bar\rho_j - \Delta_j < \rho_j < \bar\rho_j + \Delta_j, \quad \bar\rho_{j+1} - \Delta_{j+1} < \rho_{j+1} < \bar\rho_{j+1} + \Delta_{j+1},$$
$$\bar\rho_{j-1} - \Delta_{j-1} < \rho_{j-1} < \bar\rho_{j-1} + \Delta_{j-1}$$

where $\bar{\rho}_s$ are approximate values of roots ρ_s with accuracy Δ_s, then it is necessary in addition that

$$\bar{\rho}_j + \Delta_j + \varepsilon_1 < \bar{\rho}_{j+1} - \Delta_{j+1}, \quad \bar{\rho}_j - \Delta_j - \varepsilon_2 > \bar{\rho}_{j-1} + \Delta_{j-1}$$

3. Let us now consider the same problem for a system consisting of the critical system of form

$$x^{\cdot} = -\lambda y + X_0 + \sum_{l=1}^{\infty} \mu^l X_l, \quad y^{\cdot} = Y_0 + \sum_{l=1}^{\infty} \mu^l Y_l$$

and of the adjoint system of the same form as everywhere previously. For $\mu = 0$ the critical system becomes nonlinear

$$x^{\cdot} = -\lambda y + X_0, \ y^{\cdot} = Y_0 \tag{3.1}$$

whose characteristic equation contains two zero roots with a single group of solutions. We retain all assumptions about the form of functions X_l, Y_l, and X_{sl}, as well as the assumptions defined in Sect. 1, except stipulations 1° — 3°. The remark made in Sect.1 about the possibility of reducing the system to the investigated from remains valid. We make the following additional assumptions.

1°. Functions X_0 and Y_0 are of the form (1.3) in which $m_0 \geqslant 2$, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are constant coefficients, and function $Q_1 = \lambda y^2 + F(x, y)$ is positive definite throughout the investigation region $U \supset N$.

2°. There exists for (3.1) a holomorphic integral $I_1(x, y) = c^2$ of constant sign in which $I_1 = y^2 + H_1$ has all the properties of integral I defined in Sect. 2. Among these, the equation $I_1(r \cos \theta, r \sin \theta) = c^2$ has a solution for r of the form $r = c + \bar{v}_2(\theta) c^2 + \bar{v}_3(\theta) c^3 + \ldots$, which depends on a single arbitrary constant c, and $\bar{v}_k(\theta)$ are 2π- periodic functions of θ. The solutions for r is holomorphic in c and the series is absolutely convergent throughout region $U \supset N$ of steady oscillations.

We reduce the considered system, using the same transformation sequence as in Sect. 2, including substitution $\bar{r} = r + \bar{v}_2(\theta) r^2 + \bar{v}_3(\theta) r^3 + \ldots$, to the same form as in Sect. 2 with an accuracy to within the second equation of the critical system, which in this case is of the form

$$\theta^{\cdot} = \lambda \sin^2\theta + Q_0^*(r, \theta) + \sum_{l=1}^{\infty} \mu^l F_l^*(r, \theta, z_1, \ldots, z_n), \quad Q_0^*(r, \theta) = \frac{F}{\bar{r}^2}$$

where $\bar{r}$ is expressed in terms of r in conformity with the last substitution and $\lambda \sin^2\theta + Q_0^* \cdot (r, \theta) > 0$ for $r \neq 0$ by virtue of condition 1°. The equations which determine $g_1^{(l)}$ and $u_1^{(l)}(\theta)$ differ considerably in this case from those obtained above and, also, from those in [1, 2].

We have:

in the case of $m_1 < m_0$

$$u_1^{(m)}(\theta) \equiv 0,\ g_1^{(m)} = 0\ \ (m = 1, 2, \ldots, m_1 - 1) \tag{3.2}$$

$$R_1^{(1,i)}(\theta) - \lambda \sin^2\theta \frac{du_1^{(i)}}{d\theta} = g_1^{(i)}\quad (i = m_1, \ldots, m_0 - 1)$$

$$R_1^{(1,k)}(\theta) - \lambda \sin^2\theta \frac{du_1^{(k)}}{d\theta} - \sum_{s=1}^{k-m_0+1} Q_0^{*(k-s)}(\theta)\frac{du_1^{(s)}}{d\theta} = g_1^{(k)}\quad (k = m_0, m_0 + 1, \ldots)$$

where $Q_0^{*(\alpha)}$ is $Q_0^{(\alpha)}$ with the substitution of $\bar{v}_s$ for v_s ;

in the case of $m_1 = m_0$ the same system, except for the equations in the second stroke of (3.2); and

in the case of $m_1 > m_0$ the system consisting of equalitities in the first line of (3.2) and, also, of its last line for $k = m_1,\ m_1 + 1, \ldots$.

We determine $g_1^{(l)}$ in (3.2) so as to have all $u_1^{(l)}(\theta)$ 2π-periodic. To do this it is necessary and sufficient to determine $g_1^{(l)}$ by equations of the form

$$R_1^{(1,l)}(\theta) - g_1^{(l)} = \lambda \sin^2\theta f_1^{(l)}(\theta) + \Psi^{(l)}(\theta)\quad \left(\int_0^{2\pi} f_1^{(l)}(\theta)\, d\theta = 0\right) \tag{3.3}$$

where $f_1^{(l)}(\theta)$ are some 2π-periodic functions each of which satisfies the condition appearing above in parentheses. Periodic functions $u_1^{(l)}(\theta)$ are determined by the expressions

$$u_1^{(l)}(\theta) = \int_0^{\theta} f_1^{(l)}(\theta)\, d\theta$$

thus $u_1^{(l)}(0) = 0$.

For the joint determination of $g_1^{(l)}$ and $f_1^{(l)}(\theta)$ we represent functions $R_1^{(1,l)} - \Psi^{(l)}$ in the form

$$R_1^{(1,l)} - \Psi^{(l)} = a_0^{(l)} + \sum_{s=1}^{l} (a_s^{(l)} \sin s\theta + b_s^{(l)} \cos s\theta)$$

and seek functions $f_1^{(l)}$ themselves in the form of sums

$$f_1^{(l)} = \sum_{s=1}^{l-2} (c_s^{(l)} \sin s\theta + d_s^{(l)} \cos s\theta)$$

Obviously $f_1^{(l)}$ satisfy the condition formulated above. Substituting $R_1^{(1,l)} - \Psi^{(l)}$ and $f_1^{(l)}$ into (3.3) for determining the $2l - 3$ unknown $g_1^{(l)}$, $c_s^{(l)}$, and $d_s^{(l)}$ we obtain a system of $2l + 1$ equations for each l , which is compatible only when certain supplementary conditions imposed on coefficients are satisfied. These conditions for not very large l, can be readily obtained in explicit form. Further analysis is similar to that in Sect. 2.

4. Example. Let us investigate the problem of stability of the unperturbed motion $x = y = z_1 = z_2 = 0$ of a mechanical system whose perturbed motion equat-

ions are

$$x^{\cdot} = -Ay^3 + \mu\,[z_1\,(a_1x^2 + a_2y^2) + ax^5 + bx^7] + \mu^2\Phi_1 \qquad (4.1)$$
$$y^{\cdot} = Ax^3 + \mu\,[z_1\,(b_1x^2 + b_2y^2) + ay^5 + by^7] + \mu^2\Phi_2$$
$$z_1^{\cdot} = pz_1 + \mu\Phi_3 + \mu^2\Phi_4, \quad z_2^{\cdot} = qz_2 + \mu\Phi_5 + \mu^2\Phi_6$$
$$\mu > 0, \quad A > 0, \quad p < 0,\ q < 0,\ ab < 0$$
$$\Phi_1 = z_1^2\,(a_5x^2 + a_6y^2), \quad \Phi_2 = z_1^2\,(b_5x^2 + b_6y^2), \quad \Phi_3 = z_1^2\,(c_1x + c_2y)$$
$$\Phi_4 = z_1^3\,(c_3x + c_4y), \quad \Phi_5 = z_1^2\,(d_1x^2 + d_2y^2), \quad \Phi_6 = z_1^3\,(d_3x^2 + d_4y^2)$$

Subsequently the explicit form of functions Φ_s presented here is important only when estimating the dimensions of region N.

System (4.1) evidently satisfies the conditions formulated in Sect. 1, and

$$F = A\,(x^4 + y^4), \quad R = Axy\,(x^2 - y^2), \quad \varphi = {}^1/_4 \ln\,[4\,/\,(\cos 4\theta + 3)]$$

Let us investigate the stability of unperturbed motion $x = y = z_1 = z_2 = 0$ of system (4.1) using the theorem on reduction for stable motion [6]. According to Kamenkov's terminology [6] system (4.1) relates to the so-called unessential particular case, since in it the right-hand sides of the adjoint system vanish for $z_1 = z_2 = 0$, while the right-hand sides of the critical system are not identically zeros. It was shown in [6] that the problem of stability of the zero solution of the analyzed system is equivalent to the problem of stability of the zero solution of a second order system of the form

$$x^{\cdot} = -Ay^3 + hx^5 + lx^7, \quad y^{\cdot} = Ax^3 + hy^5 + ly^7, \quad h = \mu a, \quad l = \mu b$$

Two possibilities may exist.

1) $a > 0$ and $b < 0$.

By the theorem on reduction the unperturbed motion is unstable for any $\mu > 0$, while by Theorems 1 and 2 in [1] a μ^* and a region $N\,(A_0^*, \varepsilon_1^*, \varepsilon_2^*)$ such that for all $\mu < \mu^*$ and all $t \in (t_0, \infty)$ the representing point $P\,(t) \in N$, can be found if only $P\,(t_0) \in N$, where t_0 is the initial instant of time. Steady oscillations exist and are stable within region N whose dimensions with respect to coordinate ρ are determined by $\rho_1 + \varepsilon_1^*$ and $\rho_1 - \varepsilon_2^*$ in conformity with [1]. In this case

$$\rho_1 = 4\left[\frac{a}{2\pi b}\left(2\mathrm{E}\left(\frac{\sqrt{2}}{2}\right) - 3\mathrm{K}\left(\frac{\sqrt{2}}{2}\right)\right)\right]^{1/2}$$

where K and E are complete ellipitic integrals of the first and second kind. If ρ_1 is small $(a\,/\,b \to 0,\ a \neq 0)$, then for every $\mu < \mu^*$ we find that in spite of the theorem on reduction which implies instability, the deviations of $P\,(t)$ from the coordinate origin can be made small with respect to ρ by a suitable secltion of a and b, provided their initial values are small.

2) $a < 0$ and $b > 0$.

By the theorem on reduction the unperturbed motion is asymptotically stable for any $\mu > 0$. By the Theorems 1 and 2 in [1] the property defined in case 1) is also present here, but region N is different, since it contains the coordinate origin. The dimensions of N relative to coordinate ρ are also determined by ρ_1. Hence, when

$a/b \to 0$ and $a \neq 0$, the region of coordinate origin attraction contracts so that the fact of asymptotic stability looses its practical value.

The results of application of Theorems 1 and 2 [1] thus provide a definite supplement to the results obtainable with the use of the theorem on reduction. Moreover, they obviously make possible the investigation of motion not only in the equilibrium position neighborhood but, also, away from it, for instance, at considerable ρ_1.

In the above example the estimate of N was considered only in relation to coordinate ρ . The complete estimate made on the basis of Theorem 3 [1] yielded the following general system of inequalities which must be satisfied by μ and parameters of region N:

$$L_1(\rho_\alpha) + \chi_\alpha < 0, \quad L_1(\rho_\beta) - \chi_\beta > 0$$

$$\mu A_0 < \frac{1}{2\Psi_6}\left[\left(\Psi_5^2 - 4\frac{q'\Psi_6}{r_{1\alpha}}\right)^{1/2} - \Psi_5\right], \quad \mu < \frac{1}{2\pi\rho_\alpha^4[5\sigma_{10}^{(5)} + 7\rho_\alpha^2\sigma_{10}^{(7)}]}$$

where

$$\chi_\gamma = \sqrt[4]{2}A_0\rho_\gamma^2\Psi_1 + \mu\sqrt[4]{2}A_0^2 r_{1\gamma}\Psi_2 + \mu r_{2\gamma}(\Psi_{3\gamma} + \mu\Psi_{4\gamma}) \quad (\gamma = \alpha, \beta)$$

$$L_1(\rho) = g_1^{(5)}\rho^5 + g_1^{(7)}\rho^7, \qquad \rho_\alpha = \rho_j + \varepsilon_1, \quad \rho_\beta = \rho_j - \varepsilon_2,$$

$$g' = \max\{p, q\}$$

$$r_{1\gamma} = \rho_\gamma + 2\pi\mu r_{2\gamma}, \quad r_{2\gamma} = \rho_\gamma^5\sigma_{10}^{(5)} + \rho_\gamma^7\sigma_{10}^{(7)}$$

$$\sigma_{10}^{(5)} = \frac{\sqrt{2}}{A}(3|a| + |g_1^{(5)}|), \quad \sigma_{10}^{(7)} = \frac{\sqrt{2}}{A}(3\sqrt[4]{4}|b| + |g_1^{(7)}|)$$

$$\Psi_1 = {}^3/_2 S_1, \quad \Psi_2 = {}^3/_2 S_2, \quad \Psi_{3\gamma} = \sqrt[4]{2}[A_0 S_1 + 2\sqrt[4]{8}|a|r_{1\gamma}^3 + 4\sqrt[4]{2}|b|r_{1\gamma}^5]r_{1\gamma}$$

$$\Psi_{4\gamma} = \sqrt[4]{2}A_0^2 S_2 r_{1\gamma}, \quad \Psi_5 = \sqrt[4]{2}S_3 + \sqrt{2}r_{1\alpha}S_4, \quad \Psi_6 = \sqrt[4]{2}S_5 + \sqrt{2}r_{1\alpha}S_6$$

$$S_1 = |a_1| + |a_2| + |b_1| + |b_2|, \quad S_2 = |a_5| + |a_6| + |b_5| + |b_6|$$

$$S_3 = |c_1| + |c_2|, \quad S_4 = |d_1| + |d_2|, \quad S_5 = |c_3| + |c_4|, \quad S_6 = |d_3| + |d_4|$$

$$g_1^{(5)} = \frac{a}{2K(\sqrt{2}/2)}\left[3K\left(\frac{\sqrt{2}}{2}\right) - 2E\left(\frac{\sqrt{2}}{2}\right)\right], \quad g_1^{(7)} = \frac{\pi b}{16K(\sqrt{2}/2)}$$

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